Math 217 Fall 2025

Quiz 35 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V is a vector space. A linear transformation $T\colon V \to V$ is ...

   > **Solution:** A linear transformation $T\colon V \to V$ is a function such that for all $u, v \in V$ and all scalars c,
   > $$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

   (b) Let $f(x)$ be a polynomial, and let λ be a root. The *multiplicity of the root* λ is ...

   > **Solution:** The *multiplicity* of the root λ is the largest integer $m \in \mathbb{N}$ such that we can factor
   > $$f(x) = (x - \lambda)^m g(x)$$
   > for some polynomial $g(x)$.

   (c) Let λ be an eigenvalue of a linear transformation T of a finite dimensional space. The *algebraic multiplicity* of λ is ...

   > **Solution:** The *algebraic multiplicity* of λ is its multiplicity as a root of the characteristic polynomial $\chi_T(x)$ of T.

2. Let λ_1, λ_2 be *distinct* eigenvalues of a linear transformation $T : V \to V$. Show, *without using Theorem D*: if v_1 is a λ_1-eigenvector and v_2 is a λ_2-eigenvector, then (v_1, v_2) is linearly independent.

> **Solution:** Consider the equation
> $$av_1 + bv_2 = 0.$$
> Because v_1 and v_2 are eigenvectors, we have
> $$T(v_1) = \lambda_1 v_1 \quad \text{and} \quad T(v_2) = \lambda_2 v_2.$$
> Apply T to the relation $av_1 + bv_2 = 0$:
> $$T(av_1 + bv_2) = T(0) = 0.$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Using linearity of T and the eigenvector equations, this gives

$$aT(v_1) + bT(v_2) = a\lambda_1 v_1 + b\lambda_2 v_2 = 0.$$

Now we have the system

$$\begin{cases} av_1 + bv_2 = 0, \\ a\lambda_1 v_1 + b\lambda_2 v_2 = 0. \end{cases}$$

Multiply the first equation by λ_2:

$$\lambda_2 a v_1 + \lambda_2 b v_2 = 0.$$

Subtract this from the second equation:

$$(a\lambda_1 v_1 + b\lambda_2 v_2) - (\lambda_2 a v_1 + \lambda_2 b v_2) = a(\lambda_1 - \lambda_2)v_1 + b(\lambda_2 - \lambda_2)v_2 = a(\lambda_1 - \lambda_2)v_1 = 0.$$

We have $\lambda_1 \neq \lambda_2$ by assumption. Also $v_1 \neq 0$ (because it is an eigenvector). Therefore $a(\lambda_1 - \lambda_2)v_1 = 0$ implies $a = 0$.

Then from $av_1 + bv_2 = 0$ we have $bv_2 = 0$, and since $v_2 \neq 0$, it follows that $b = 0$. Thus the only linear combination giving 0 is the trivial one, $a = b = 0$.

Hence v_1 and v_2 are linearly independent.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a diagonal matrix.

Solution: TRUE. A 2×2 matrix is diagonal if all its off-diagonal entries are 0. Here the matrix is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

whose off-diagonal entries are both 0, so it is diagonal (the diagonal entries are allowed to be 0).

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is a diagonal matrix.

Solution: TRUE. The matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

has zeros in the off-diagonal positions, so it is diagonal.

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is a diagonal matrix.

Solution: TRUE. Again, the off-diagonal entries of

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

are 0, so by definition the matrix is diagonal.

4. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a diagonalizable matrix.

Solution: FALSE.
Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = \lambda^2,$$

so the only eigenvalue is $\lambda = 0$ with algebraic multiplicity 2.
To find the eigenspace for $\lambda = 0$, we solve $A\vec{v} = 0$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad y = 0.$$

So eigenvectors have the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$, and the eigenspace is

$$E_0 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\},$$

which is 1-dimensional.
Thus the geometric multiplicity of 0 is 1, which is strictly less than its algebraic multiplicity 2. Therefore A is not diagonalizable.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ is a diagonalizable matrix.

Solution: TRUE.
Let

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(B - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 0 & 1 - \lambda \end{bmatrix} = (-\lambda)(1 - \lambda) - 0 = -\lambda(1 - \lambda).$$

The eigenvalues are $\lambda = 0$ and $\lambda = 1$, which are distinct.
For a 2×2 matrix with two distinct eigenvalues, the eigenspaces automatically have dimension 1 each, so we get two linearly independent eigenvectors. Thus B has a basis of eigenvectors and is diagonalizable.
(Alternatively, one can explicitly find eigenvectors for $\lambda = 0$ and $\lambda = 1$ and check that they are linearly independent.)

(c) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ is a diagonalizable matrix.

Solution: TRUE.

Let

$$C = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(C - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 1 \\ 0 & 3 - \lambda \end{bmatrix} = (2 - \lambda)(3 - \lambda).$$

So the eigenvalues are $\lambda = 2$ and $\lambda = 3$, which are distinct.

Again, a 2×2 matrix with two distinct eigenvalues is diagonalizable, since it has two linearly independent eigenvectors. Therefore C is diagonalizable.